Harmonic Tidal Theory

An investigation of discrepancies between different implementations of the harmonic method

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Outline

- Relevance of Tides
- History: Harmonic method
- Harmonic Listings
- Modulation - a variation of the harmonic method
- Compare two implementations
- Discussion of Discrepancies
- Summary and Conclusion
Relevance of Tides

- Running Aground: Tidal Levels, ML, HAT, LAT etc
- Optimising Fuel Usage
- Tidal Energy
- Revisit Tidal Analysis
1687 Isaac Newton, Principia Mathematica
- Superposition of tides
- Diurnal inequality

1776 Laplace
- Surpassed Newton's work
- Eliminating the concept of bulges
- Horizontal tidal force
- "astres fictifs"

1822 Fourier
1872 Lord Kelvin's First Tidal Prediction Machine
1878 Hill’s Lunar Orbit
1883 Sir G. H. Darwin harmonic tidal constituents.
1905 Brown’s Lunar Orbit
Brown's - Mean Lunar Orbit 1905

- 655 terms for Lunar Longitude
- 300 terms for Lunar Latitude
- Accurate to within 0.1 seconds of arc
- Periodic terms all of known frequency
AT Doodson 1921

Re-calculated the tidal force using Brown's Lunar Orbit
Included all significant solar and lunar orbital effects
400 pure harmonic frequencies - none were modulated as Darwins were
Each with their own amplitude and phase
Ranges from 1 per 18.6 years to 3 times per day.
Harmonic Listings

- Pure Harmonic Listing
  - A.T. Doodson 1921
  - Cartwright & Tayler 1971
  - Buellesfield 1985
  - Tamura 1987
  - Xi Qin-wen 1987

- Listings which include modulation
  - Lord Darwin 1883
  - UKHO Admiralty Manual of Tides 1941 & numerous later publications
  - UKHO Simplified Method
  - IHO Tidal Committee 2003
Tide = ML + Σ_{all i} (H_i \cdot D_i \sin (w_i t + g_i + d_i))
Modulation

**Time Domain**

- Real Tide

\[ \sin \left( \frac{w_{SD} t}{2} \right) \cdot \cos \left( \frac{w_{ft}}{2} \right) \]

**Frequency Domain**

- Tidal Raising Force
- Ocean Response
- Tidal Spectrum

\[ \sin (w_{1} t) + \sin (w_{2} t) \]

\[ 2 \sin \left( \frac{A + B}{2} \right) \cdot \cos \left( \frac{A - B}{2} \right) = \sin A + \sin B \]
Lunar Orbit

- Lunar Orbital Complex
- Elliptical orbit ~ 5.5%
- Precession of line of the apsides: perigee & apogee
- Period 8.5 years
- Precession of Orbital Plane
- Period 18.6 years
Precession of Orbital Plane

- 18.6 years
- Changes the amplitude and phase of constituents
- "Nodal Modulation"
- Can be calculated with formulae or by adding constituents
- But both approaches should be equivalent
- How to compare
How to compare

1. Compare the published formulae with published constituent listings

2. Use tidal prediction software itself
Pure Harmonic Listing

\[ h = \left(\frac{1}{\text{imax}}\right) \sum I_i \cos (w_{SD} + j \cdot N + \Phi_i) t \quad \ldots \ldots \ldots (2) \]
Modulation Formulae

Time Domain

Modulation Formulae

\[ h = f_{\text{of } x} \cdot \cos (w_{\text{SD}} \cdot t + u_{\text{of } x} \cdot \pi / 180 + \Phi_i) \] .............(3)

Modulation includes both AM and FM

\[ u_{\text{MSf}} = 2.14 \sin N \]

\[ f_{\text{MSf}} = 1.0004 - 0.0373 \cos N + 0.0002 \cos 2N \]
Comparison using Formulae

- Method
  - Enter previous two equations into a spreadsheet
  - Use normalised values for w
  - Calculate at 10,000 dates over 18 years
  - Calculate RMS error
<table>
<thead>
<tr>
<th>ETC Listing</th>
<th>Nodal Formulae</th>
<th>Normalised RMS Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>M2</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2555355, 0.00047</td>
<td>$u_{M2} = -2.14 \sin N$</td>
<td>0.01</td>
</tr>
<tr>
<td>2555457, 0.03386</td>
<td>$f_{M2} = 1.0004 - 0.0373 \cos N + 0.0002 \cos 2N$</td>
<td></td>
</tr>
<tr>
<td>2555555, 0.90812</td>
<td>[13]</td>
<td></td>
</tr>
<tr>
<td>2555355, 0.00047</td>
<td>$u_{M2} = -2.14 \sin N$</td>
<td>0.02</td>
</tr>
<tr>
<td>2555457, 0.03386</td>
<td>$f_{M2} = 1.0007 - 0.0373 \cos N + 0.0002 \cos 2N$</td>
<td></td>
</tr>
<tr>
<td>2555555, 0.90812</td>
<td>[19]</td>
<td></td>
</tr>
<tr>
<td><strong>O1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1455356, 0.00218</td>
<td>$u_{O1} = 10.80 \sin N - 1.34 \sin 2N + 0.19 \sin 3N$</td>
<td>0.03</td>
</tr>
<tr>
<td>1455454, 0.07105</td>
<td>$f_{O1} = 1.0089 + 0.1871 \cos N - 0.0147 \cos 2N + 0.0014 \cos 3N$</td>
<td></td>
</tr>
<tr>
<td>1455554, 0.37689</td>
<td>[13]</td>
<td></td>
</tr>
<tr>
<td>1455356, 0.00218</td>
<td>$u_{O1} = 10.80 \sin N - 1.34 \sin 2N + 0.19 \sin 3N$</td>
<td>0.62</td>
</tr>
<tr>
<td>1455454, 0.07105</td>
<td>$f_{O1} = 1.0176 + 0.1871 \cos N - 0.0147 \cos 2N$</td>
<td></td>
</tr>
<tr>
<td>1455554, 0.37689</td>
<td>[19]</td>
<td></td>
</tr>
<tr>
<td><strong>K1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1655454, 0.01050</td>
<td>$u_{K1} = -8.86 \sin N + 0.68 \sin 2N - 0.07 \sin 3N$</td>
<td>0.01</td>
</tr>
<tr>
<td>1655556, 0.53007</td>
<td>$f_{K1} = 1.0060 + 0.1150 \cos N - 0.0088 \cos 2N + 0.0006 \cos 3N$</td>
<td></td>
</tr>
<tr>
<td>1655656, 0.07182</td>
<td>[13]/[19]</td>
<td></td>
</tr>
<tr>
<td>1655754, 0.00154</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tidal Constituent</td>
<td>ETC Listing</td>
<td>Nodal Formulae</td>
</tr>
<tr>
<td>-------------------</td>
<td>------------------------------</td>
<td>-------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>MSf</td>
<td>0735455, 0.00098</td>
<td>$u$ of MSf = $-u$ of M2 and $f$ of MSf = $f$ of M2 [13]/[19]</td>
</tr>
<tr>
<td></td>
<td>0735555, 0.01370</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0735657, 0.00088</td>
<td></td>
</tr>
<tr>
<td>MP1</td>
<td>1475454, 0.00014</td>
<td>$u$ and $f$ same as $M_2$ [19]</td>
</tr>
<tr>
<td></td>
<td>1475556, 0.00491</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1475654, 0.00107</td>
<td></td>
</tr>
<tr>
<td>MP1</td>
<td>1475454, 0.00014</td>
<td>$u$ and $f$ same as $O_1$ [13]</td>
</tr>
<tr>
<td></td>
<td>1475556, 0.00491</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1475654, 0.00107</td>
<td></td>
</tr>
</tbody>
</table>
Comparison using Tidal Prediction Software

- **TotalTide**
  - Uses UKHO IHO constituent harmonic list
  - Implicitly uses nodal modulation formulae - UKHO / IHO documents.

- **GeoTide**
  - Uses pure summation of harmonic list from Doodson
  - as published by The Earth Tides Commission at International Association of Geodesy
Method

- Create artificial harmonic constants
- Set constituent under study to 100m
- At twenty random dates
- Predicted values were then analysed in a spreadsheet
- Advantages
  - Independent of values of frequencies
  - Independent of software implementation
<table>
<thead>
<tr>
<th>TotalTide Constituent</th>
<th>ETC Listing</th>
<th>Normalised RMS Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M2</td>
<td>2555355, 2555457, 2555555</td>
<td>0.6</td>
</tr>
<tr>
<td>S2</td>
<td>2735555</td>
<td>0.0</td>
</tr>
<tr>
<td>K1</td>
<td>1655454, 1655556, 1655656, 1655754</td>
<td>0.1</td>
</tr>
<tr>
<td>O1</td>
<td>1455356, 1455454, 1455554</td>
<td>0.5</td>
</tr>
<tr>
<td>Mnum</td>
<td>0636457, 0636555, 0636657</td>
<td>3.4</td>
</tr>
<tr>
<td>Mm</td>
<td>0654457, 0654555, 0654657</td>
<td>0.3</td>
</tr>
<tr>
<td>MSf</td>
<td>0735455, 0735555, 0735657</td>
<td>5.6</td>
</tr>
<tr>
<td>Mf</td>
<td>0755555, 0755655, 0755755, 0755857</td>
<td>0.8</td>
</tr>
<tr>
<td>2Q1</td>
<td>1257454, 1257554</td>
<td>1.2</td>
</tr>
<tr>
<td>sig1</td>
<td>1275454, 1275554</td>
<td>1.3</td>
</tr>
<tr>
<td>Q1</td>
<td>1356356, 1356454, 1356554</td>
<td>0.8</td>
</tr>
<tr>
<td>rho1</td>
<td>1374454, 1374554</td>
<td>0.8</td>
</tr>
<tr>
<td>MS1</td>
<td>1465444, 1465544</td>
<td>139.4</td>
</tr>
<tr>
<td>MP1</td>
<td>1475454, 1475556, 1475654</td>
<td>15.8</td>
</tr>
<tr>
<td>NO1</td>
<td>1556556, 1556656</td>
<td>0.9</td>
</tr>
<tr>
<td>M1</td>
<td>1556556, 1556656</td>
<td>46.5</td>
</tr>
</tbody>
</table>

*Table 1. A Comparison of TotalTide and GeoTide on a constituent by constituent basis*
MSf

- 5% Discrepancy
- ETC Listing: It's a Frequency Modulated tidal potential
- MSf is both a compound (M2-S2) and pure components
- In UKHO is modulated like -M2

<table>
<thead>
<tr>
<th>Mmm</th>
<th>(MMH)</th>
<th>Mm</th>
<th>MSf</th>
<th>MSo</th>
<th>SM</th>
<th>Mf</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.471321</td>
<td>0.656533</td>
<td>0.544375</td>
<td>1.015896</td>
<td>1.015896</td>
<td>1.015896</td>
<td>1.008033</td>
</tr>
<tr>
<td>Z AXA ZZZ</td>
<td>Z AXA ZZZ</td>
<td>Z AZY ZZZ</td>
<td>Z BXZ ZZZ</td>
<td>Z BXZ ZZZ</td>
<td>Z BXZ ZZZ</td>
<td>Z BZZ ZZZ</td>
</tr>
<tr>
<td>0.717555</td>
<td>0.000260</td>
<td>0.725565</td>
<td>0.000913</td>
<td>0.735455</td>
<td>0.000980</td>
<td>0.735555</td>
</tr>
<tr>
<td>95</td>
<td>98</td>
<td>105</td>
<td>106</td>
<td>107</td>
<td>110</td>
<td>119</td>
</tr>
</tbody>
</table>
M1/NO1

- M1 46%
- NO1 0.9%
- Same frequency
- But IHO Listing has 3 M1s
- ETC
- NO1 is not small
MS1:

- ETC Listing
- IHO Listing
- TotalTide very large modulation
- Fortunately very small ~mm
MP1

- 15% difference
- ETC Listing
- 5mm
- The sidebands are same phase
- ETC is actually referring to Tau1 which is of identical frequency.
- Tau1 is not listed in TT
Results Summary: Major Constituents

M2
- Predictor Comparison to 0.6%
- Formulae comparison agrees to 0.01% \(^{[13]}\) or 0.02% \(^{[19]}\)

S2
- The agreement is <0.01% in both tests

K1
- Predictor Comparison to 0.1%
- Formulae comparison agrees to 0.01%

O1
- Predictor Comparison to 0.5%
- Formulae comparison agrees to 0.03%\(^{[13]}\) or 0.62%\(^{[19]}\)

Depending on which published formulae is used ref 13 or ref 19.
13 Admiralty Manual of Tides AT Doodson 1941
19 IHO Tidal Constituents Table Vth IHO Tidal Committee 2003
Results Summary: Minor Constituents

MP1

Also ETC refers to Tau1 which is of almost identical frequency.
Also it suffers from an inconsistency in published nodal modulation formulae.

M1

3 components with identical name M1 are listed by IHO and it is not clear to which TotalTide refers.
The ETC component tallies with NO1 of the same frequency and agrees to within 0.9%

MSf

Is used to refer both to the astronomical constituent and the compound constituent S2-M2. Since these have different nodal formulae this could well account for the discrepancy.

MS1

ETC refers to a pure components while MS1, we probably refers to a compound component of similar frequency. However, MS1 in TotalTide seems to have a very large level of Nodal Modulation
Conclusions

- Two comparison of two different variations on the harmonic method was attempted.

- On M2, S2, K1, O1 agreement was very good.

- The majority of minor constituents tested showed acceptable level of agreement but 3 discrepancies were noted (>4%) on minor constituents, which were explained mainly by ID issues.

- MSf may have two different types of nodal modulation depending upon the type of tide. ~mm

- Some inconsistencies in the formulae were also found in the literature which should be further tracked down.
Additional Notes

073555 (MSf)

It is said in the admiralty manual of tides that this is a shallow water compound tide which probably masks the astronomical component. But is this the case. Could be interpreted as pure or as M2-S2 273555-255555 amplitudes are 0.9 and 0.1 respectively. In the case of compound tide it also requires the coefficient of non-linearity. Perhaps ETC's 0735555 would correspond better to SM as listed in IHO but not in TotalTide.

155655 (M1/NO1)

It is surprising that agreement is better with NO1 rather than M1. However M1 appears 3 times: one entry has only a different phase - so that's two different frequencies. The third entry is the same as NO1 but varying in nodal modulation formulae. It is not clear which M1 is referred to. ETC's 155655 would correspond therefore to NO1.

In US Coastguards book M1 is listed at two frequencies but with three origins one of which is a compound tide.

1465544 (MS1)

ETC's 1465544 is probably not MS1 which is listed with a very slightly different frequency. MS1 must therefore be a compound having different nodal modulation.

147555 (MP1)

ETC's 147555 is probably Tau rather than MP1 of the same frequency. Unfortunately Tau is not listed in TotalTide but has an amplitude of 5mm. The comparison was actually made with MP1.
Doodson Notation

7 digit number e.g. 2544667

Constituent ID = \( d_1 \ d_2 \ d_3 \ d_4 \ d_5 \ d_6 \ d_7 \)

Apart from first digit - 5 must be subtracted.

Phase = \( 90^\circ \times (d_7 - 5) \) i.e. sin or cosine terms

Frequency = \( 0.96d_1 + 0.036d_2 + 0.0027d_3 + 0.00031d_4 + 0.00015d_5 + 0.00000013d_6 \)

Daily       Month       Year     8 y        18y        26,000y

Lunar Precession of Apsides       Lunar Precession of Plane       Solar Precession of Obliquity